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WEIGHTED WEAK TYPE $(1, 1)$ ESTIMATES FOR OSCILLATORY SINGULAR INTEGRALS

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ABSTRACT. We consider the A_1 -weights and prove the weighted weak type $(1, 1)$ inequalities for certain oscillatory singular integrals.

1. INTRODUCTION

Let $K \in C^1(\mathbf{R}^n \setminus \{0\})$ satisfy

$$(1.1) \quad |K(x)| \leq c|x|^{-n}, \quad |\nabla K(x)| \leq c|x|^{-n-1};$$

$$(1.2) \quad \int_{a < |x| < b} K(x) dx = 0 \quad \text{for all } a, b \ (0 < a < b).$$

The smallest constant for which (1.1) holds will be denoted by $C(K)$.

We consider an oscillatory singular integral operator:

$$T(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y) dy = \lim_{\epsilon \rightarrow 0} \int_{|x-y| > \epsilon} e^{iP(x,y)} K(x-y) f(y) dy,$$

initially defined for $f \in \mathfrak{S}(\mathbf{R}^n)$ (the Schwartz space), where P is a real-valued polynomial:

$$(1.3) \quad P(x, y) = \sum_{|\alpha| \leq M, |\beta| \leq N} a_{\alpha\beta} x^\alpha y^\beta.$$

The following results are known.

Theorem A. (Ricci-Stein [9]) *Let $1 < p < \infty$. Then, T is bounded on $L^p(\mathbf{R}^n)$ with the operator norm bounded by a constant depending only on the total degree of P , $C(K)$, p and the dimension n .*

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Theorem B. (Chanillo-Christ [2]) *The operator T is bounded from $L^1(\mathbf{R}^n)$ to the weak $L^1(\mathbf{R}^n)$ space:*

$$\sup_{\lambda>0} \lambda |\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}| \leq c \|f\|_{L^1},$$

with a constant c depending only on the total degree of P , $C(K)$ and the dimension n .

See also [1] and [3] for the weighted weak type $(1, 1)$ estimates for convolution operators with oscillating kernels.

Let w be a locally integrable positive function on \mathbf{R}^n . We say that $w \in A_1$ if there is a constant c such that

$$(1.4) \quad M(w)(x) \leq cw(x) \quad \text{a.e.}$$

where M denotes the Hardy-Littlewood maximal operator. The smallest constant for which (1.4) holds will be denoted by $C_1(w)$.

In this note we shall prove that T is bounded from L_w^1 to $L_w^{1,\infty}$ (the weak L_w^1 space) for $w \in A_1$:

Theorem. *There exists a constant c depending only on the total degree of P , $C(K)$, $C_1(w)$ and the dimension n such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

where $w(E) = \int_E w(x) dx$ and $\|f\|_{L_w^1} = \int |f(x)|w(x) dx$.

The theorem will be proved by a double induction, as in [9] and [2]. Let P be a polynomial as in (1.3). We assume that there exists a multi-index α such that $|\alpha| = M$ and $a_{\alpha\beta} \neq 0$ for some β . We write

$$(1.5) \quad P(x, y) = \sum_{|\alpha| \leq M} x^\alpha Q_\alpha(y)$$

and define

$$L = \max\{\deg(Q_\alpha) : Q_\alpha \neq 0, |\alpha| = M\}.$$

Then $0 \leq L \leq N$. We assume that $L \geq 1$ and

$$\max_{\substack{|\alpha|=M \\ |\beta|=L}} |a_{\alpha\beta}| = 1.$$

Under this assumption on a polynomial P , we define

$$T_\infty(f)(x) = \int_{|x-y|>1} e^{iP(x,y)} K(x-y) f(y) dy.$$

To prove Theorem, we shall use the following result in the induction.

Proposition. *Let $\eta > 0$. There exists a constant c depending only on the total degree of P , η and the dimension n such that if $C(K)$, $C_1(w) \leq \eta$, then*

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_\infty(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1}.$$

Remark 1. By Theorem and the extrapolation theorem of Rubio de Francia, we get the L_w^p -boundedness of T for all $p \in (1, \infty)$ and all $w \in A_p$, where L_w^p is the space of all those measurable functions f which satisfy $\|f\|_{L_w^p} = (\int |f(x)|^p w(x) dx)^{1/p} < \infty$ and A_p denotes the weight class of Muckenhoupt.

We shall give the outlines of the proofs of Theorem and Proposition in Sections 2 and 4, respectively. Our proof of Proposition is based on the techniques used in Christ [5] to prove the weak type $(1, 1)$ estimates for rough operators (see also Christ [6], Christ-Rubio [7] and Sato [10]). We also use the geometrical argument of Chanillo-Christ [2]. We have to prove a key estimate (Lemma 7 in §5) in the unweighted case in order to apply the method of Vargas [11] involving an interpolation with change of measure. To prove Lemma 7, we need a geometrical result for polynomials (Lemma 5 in §5). We shall prove Lemma 5 in §7 by using the results and the arguments appearing in the proof of Chanillo-Christ [2, LEMMA 4.1].

Finally, we note that in this paper, the constants with the same notation are not necessarily the same at each occurrence.

2. OUTLINE OF PROOF OF THEOREM

To apply the induction argument of [9] we need some preparation. We may assume that $M \geq 1$ and $N \geq 1$; otherwise Theorem reduces to a well-known fact that the operator:

$$A(f)(x) = \text{p. v.} \int K(x-y)f(y) dy$$

is bounded from L_w^1 to $L_w^{1,\infty}$ (see, for example, [8]).

We write a polynomial in (1.3) as follows:

$$P(x, y) = \sum_{j=0}^M \sum_{|\alpha|=j} x^\alpha Q_\alpha(y) = \sum_{j=0}^M P_j(x, y),$$

say. We further decompose P_j as follows:

$$P_j(x, y) = \sum_{t=0}^N \sum_{\substack{|\alpha|=j \\ |\beta|=t}} a_{\alpha\beta} x^\alpha y^\beta = \sum_{t=0}^N P_{jt}(x, y),$$

say. For $j = 1, 2, \dots, M$ and $k = 0, 1, \dots, N$, define

$$(2.1) \quad R_{jk}(x, y) = \sum_{s=0}^{j-1} P_s(x, y) + \sum_{t=0}^k P_{jt}(x, y).$$

Note that $R_{jN} = \sum_{s=0}^j P_s$ ($j = 1, 2, \dots, M$).

For $j = 1, 2, \dots, M$ and $k = 0, 1, \dots, N$, we consider the following propositions.

Proposition $A(j, k)$. *Let $\eta > 0$. There exists a constant c depending only on η , j , N and the dimension n such that if $C(K)$, $C_1(w) \leq \eta$ and if R_{jk} is a polynomial as in (2.1), then*

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_{jk}(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

where

$$T_{jk}(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} e^{iR_{jk}(x,y)} K(x-y) f(y) dy.$$

Then, Theorem follows from Proposition $A(M, N)$. We shall prove it by double induction. We first note that $A(1, 0)$ follows from the $L_w^1 - L_w^{1,\infty}$ boundedness of the operator A .

Next, we observe that if $M \geq 2$ and if $A(j, N)$ ($1 \leq j \leq M-1$) is true, so is $A(j+1, 0)$ since

$$R_{j+1,0}(x, y) = R_{jN}(x, y) + \sum_{|\alpha|=j+1} a_{\alpha 0} x^\alpha$$

and hence $|T_{j+1,0}(f)(x)| = |T_{jN}(f)(x)|$. Thus, to complete the induction starting from $A(1, 0)$ and arriving at $A(M, N)$, it suffices to prove $A(j, k+1)$ by assuming $A(j, k)$ ($0 \leq k < N$, $1 \leq j \leq M$). To achieve this, put $R = R_{j,k+1}$, $R_0 = R_{jk}$, $T_{j,k+1} = S$. We note that

$$R(x, y) = R_0(x, y) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} x^\alpha y^\beta.$$

We may assume $C_{jk} = \max_{|\alpha|=j, |\beta|=k+1} |a_{\alpha\beta}| \neq 0$. Then, by a suitable dilation we may assume $C_{jk} = 1$. This can be seen as follows. We first note that, for $a > 0$,

$$S(f)(ax) = \text{p. v.} \int e^{iR(ax, ay)} K_a(x-y) f(ay) dy,$$

where $K_a(x) = a^n K(ax)$. Assume the boundedness of S for the case $C_{jk} = 1$. Then, choosing a to satisfy $a^{j+k+1} C_{jk} = 1$, and using the dilation invariance of both the class A_1 and the class of the kernels satisfying (1.1) and (1.2), we get

$$\begin{aligned} w(\{x \in \mathbf{R}^n : |S(f)(x)| > \lambda\}) &= w_a(\{x \in \mathbf{R}^n : |S(f)(ax)| > \lambda\}) \\ &\leq c \lambda^{-1} \int |f(ax)| a^n w(ax) dx \\ &= c \lambda^{-1} \|f\|_{L_w^1}. \end{aligned}$$

We split the kernel K as $K = K_0 + K_\infty$, where $K_0(x) = K(x)$ if $|x| \leq 1$ and $K_\infty(x) = K(x)$ if $|x| > 1$. Assuming $C_{jk} = 1$, we consider the corresponding splitting $S = S_0 + S_\infty$:

$$S_0(f)(x) = \text{p. v.} \int e^{iR(x,y)} K_0(x-y) f(y) dy,$$

$$S_\infty(f)(x) = \int e^{iR(x,y)} K_\infty(x-y) f(y) dy.$$

In the next section, we shall prove

$$(2.2) \quad \sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |S_0(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1},$$

while by Proposition we have

$$(2.3) \quad \sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |S_\infty(f)(x)| > \lambda\}) \leq c \|f\|_{L_w^1}.$$

Combining (2.2) and (2.3), we shall complete the proof of $A(j, k+1)$, which will finish the proof of Theorem.

3. ESTIMATE FOR S_0

In this section, we shall prove, under the assumption made in §2, that if $C(K)$, $C_1(w) \leq \eta$ ($\eta > 0$), then S_0 satisfies (2.2) with a constant c depending only on j , N , η and n .

First, we shall prove

$$(3.1) \quad w(\{x \in B(0, 1) : |S_0(f)(x)| > \lambda\}) \leq c \lambda^{-1} \int_{|y| < 2} |f(y)| w(y) dy,$$

where $B(x, r)$ denotes the closed ball with center x and radius $r > 0$.

Lemma 1. *Let $w \in A_1$. Let T be an operator of the form:*

$$T(f)(x) = \text{p. v.} \int_{\mathbf{R}^n} K(x, y) f(y) dy = \lim_{\epsilon \rightarrow 0} \int_{|x-y| > \epsilon} K(x, y) f(y) dy \quad \text{a. e.}$$

for $f \in L_w^1$, where the kernel K satisfies $|K(x, y)| \leq c_0 |x - y|^{-n}$. For $\epsilon > 0$, put

$$T_\epsilon(f)(x) = \text{p. v.} \int_{|x-y| < \epsilon} K(x, y) f(y) dy.$$

Suppose

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}) \leq c_w \|f\|_{L_w^1}.$$

Then, there exists a constant c depending only on the dimension n such that

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbf{R}^n : |T_\epsilon(f)(x)| > \lambda\}) \leq c(c_w + c_0 C_1(w)) \|f\|_{L_w^1}.$$

Proof. The proof is similar to that of LEMMA in [9, p. 187]. We shall prove

$$(3.2) \quad w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f)(x)| > \lambda\}) \leq (2c_w + cc_0 C_1(w)) \lambda^{-1} \int_{|y-h| < 5\epsilon/4} |f(y)| w(y) dy$$

uniformly in $h \in \mathbf{R}^n$. Integrating both sides of the inequality in (3.2) with respect to h , we get the conclusion of Lemma 1.

Split f into 3 pieces: $f = f_1 + f_2 + f_3$, where $f_1(y) = f(y)$ if $|y - h| < \epsilon/2$, $f_1(y) = 0$ otherwise; $f_2(y) = f(y)$ if $\epsilon/2 \leq |y - h| < 5\epsilon/4$, $f_2(y) = 0$ otherwise; $f_3(y) = f(y)$ if $|y - h| \geq 5\epsilon/4$, $f_3(y) = 0$ otherwise. Note that if $|x - h| \leq \epsilon/4$, then $T_\epsilon(f_1)(x) = T(f_1)(x)$; since $|y - h| < \epsilon/2$ and $|x - h| \leq \epsilon/4$ imply $|x - y| < \epsilon$. So by the assumption on T , we have

$$\begin{aligned} w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f_1)(x)| > \lambda\}) &= w(\{x \in B(h, \epsilon/4) : |T(f_1)(x)| > \lambda\}) \\ &\leq w(\{x : |T(f_1)(x)| > \lambda\}) \\ &\leq c_w \lambda^{-1} \|f_1\|_{L_w^1} \\ &\leq c_w \lambda^{-1} \int_{|y-h| < 5\epsilon/4} |f(y)| w(y) dy. \end{aligned}$$

Next, if $|x - h| \leq \epsilon/4$ and $\epsilon/2 \leq |y - h| < 5\epsilon/4$, then $\epsilon/4 \leq |x - y| < 3\epsilon/2$, and so

$$|T_\epsilon(f_2)(x)| \leq cc_0 \epsilon^{-n} \int_{|y-h| < 5\epsilon/4} |f_2(y)| dy.$$

Hence, by Chebyshev's inequality,

$$\begin{aligned} w(\{x \in B(h, \epsilon/4) : |T_\epsilon(f_2)(x)| > \lambda\}) &\leq cc_0 \lambda^{-1} w(B(h, \epsilon/4)) \epsilon^{-n} \int_{|y-h| < 5\epsilon/4} |f_2(y)| dy \\ &\leq cc_0 C_1(w) \lambda^{-1} \int_{|y-h| < 5\epsilon/4} |f(y)| w(y) dy. \end{aligned}$$

Finally, if $|x - h| \leq \epsilon/4$ and $|y - h| \geq 5\epsilon/4$, then $|x - y| \geq \epsilon$, and so $T_\epsilon(f_3)(x) = 0$. Combining these, we get (3.2). This completes the proof of Lemma 1.

Now we turn to the proof of (3.1). If $|x| \leq 1$ and $|y| \leq 2$, then

$$\left| \exp(iR(x, y)) - \exp\left(i\left(R_0(x, y) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} y^{\alpha+\beta}\right)\right) \right| \leq c|x - y|,$$

where c depends only on k, j and n .

Hence, if $|x| \leq 1$,

$$|S_0(f)(x)| \leq \left| U\left(\exp\left(i \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} y^{\alpha+\beta}\right) f(y)\right)(x) \right| + cI(f)(x),$$

where

$$\begin{aligned} U(f)(x) &= \text{p. v.} \int e^{iR_0(x, y)} K_0(x - y) f(y) dy, \\ I(f)(x) &= \int_{|x-y| < 1} |x - y|^{-n+1} |f(y)| dy. \end{aligned}$$

Note that $U(f)(x) = U(f\chi_{B(0,2)})(x)$, $I(f)(x) = I(f\chi_{B(0,2)})(x)$ if $|x| < 1$. By the induction hypothesis $A(j, k)$ and Lemma 1, we see that U is bounded from L_w^1 to $L_w^{1,\infty}$. On the other hand, since

$$\begin{aligned} \int_{|x-y|<1} |x-y|^{-n+1} w(x) dx &= \sum_{j \leq 0} \int_{2^{j-1} \leq |x-y| \leq 2^j} |x-y|^{-n+1} w(x) dx \\ &\leq c \sum_{j \leq 0} 2^j 2^{-jn} \int_{|x-y| \leq 2^j} w(x) dx \\ &\leq cM(w)(y), \end{aligned}$$

by Chebyshev's inequality we have

$$\begin{aligned} w(\{x \in B(0, 1) : I(f)(x) > \lambda\}) &\leq \lambda^{-1} \int_{|y|<2} \left(\int_{|x-y|<1} |x-y|^{-n+1} w(x) dx \right) |f(y)| dy \\ &\leq cC_1(w) \lambda^{-1} \int_{|y|<2} |f(y)| w(y) dy. \end{aligned}$$

Combining these results, we get (3.1).

Similarly we can prove

$$(3.3) \quad w(\{x \in B(h, 1) : |S_0(f)(x)| > \lambda\}) \leq c\lambda^{-1} \int_{|y-h|<2} |f(y)| w(y) dy,$$

where c is independent of $h \in \mathbf{R}^n$. To see this, we first note that

$$S_0(f)(x+h) = \text{p. v.} \int e^{iR(x+h, y+h)} K_0(x-y) f(y+h) dy$$

and

$$R(x+h, y+h) = R_1(x, y, h) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} x^\alpha y^\beta.$$

We can apply the induction hypothesis $A(j, k)$ to the operator

$$\text{p. v.} \int e^{iR_1(x, y, h)} K(x-y) f(y) dy$$

to get its boundedness from L_w^1 to $L_w^{1,\infty}$. Thus, by the same argument that leads to (3.1) we get

$$\begin{aligned} w(\{x \in B(h, 1) : |S_0(f)(x)| > \lambda\}) &= \tau_h w(\{x \in B(0, 1) : |S_0(f)(x+h)| > \lambda\}) \\ &\leq c\lambda^{-1} \int_{|y|<2} |f(y+h)| w(y+h) dy \\ &\leq c\lambda^{-1} \int_{|y-h|<2} |f(y)| w(y) dy, \end{aligned}$$

where $\tau_h w(x) = w(x+h)$, and we have used the translation invariance of the class A_1 . Integrating both sides of the inequality (3.3) with respect to h , we get (2.2).

4. OUTLINE OF PROOF OF PROPOSITION

We may assume $f \in \mathfrak{S}(\mathbf{R}^n)$. By Calderón-Zygmund decomposition at height $\lambda > 0$ we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g, b such that

$$(4.1) \quad f = g + b;$$

$$(4.2) \quad \lambda \leq |Q|^{-1} \int_Q |f| \leq c\lambda;$$

$$(4.3) \quad v(\cup Q) \leq c_v \|f\|_{L_v^1} / \lambda \quad \text{for all } v \in A_1;$$

$$(4.4) \quad \|g\|_\infty \leq c\lambda;$$

$$(4.5) \quad \|g\|_{L_v^1} \leq c_v \|f\|_{L_v^1} \quad \text{for all } v \in A_1;$$

$$(4.6) \quad b = \sum_Q b_Q;$$

$$(4.7) \quad \text{supp}(b_Q) \subset Q;$$

$$(4.8) \quad \int b_Q = 0;$$

$$(4.9) \quad \|b_Q\|_{L^1} \leq c\lambda|Q|.$$

Remark 2. In this note we do not use (4.8).

Let a polynomial P be as in Proposition. We assume, as we may, that $M \geq 1$ as in the outline of the proof of Theorem in §2. We write P as in (1.5). Then, let $q(y) = \sum_{|\beta| \leq L} c_\beta y^\beta$ be the coefficient of x_1^M . By a rotation of coordinates and a normalization, and by discarding a negligible difference, we see that to prove Proposition we may study T_∞ assuming $\max_{|\beta|=L} |c_\beta| = 1$; in this case the condition $\max_{|\alpha|=M, |\beta|=L} |a_{\alpha\beta}| = 1$ may not hold (see [2, p. 151] and Sublemma 2 in §7).

We pick a non-negative $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\text{supp}(\varphi) \subset \{1/2 \leq |x| \leq 2\}, \quad \sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \quad \text{if } |x| \geq 1.$$

Put $K_j(x, y) = \varphi(2^{-j}(x - y))K_\infty(x, y)$, where $K_\infty(x, y) = e^{iP(x, y)}K_\infty(x - y)$ ($K_\infty(x)$ is as in §2) and decompose $K_\infty(x, y)$ as $K_\infty(x, y) = \sum_{j=0}^{\infty} K_j(x, y)$.

Define

$$V_j(f)(x) = \int K_j(x, y) f(y) dy \quad \text{for } j \geq 0$$

and put

$$V(f)(x) = \sum_{j=1}^{\infty} V_j(f)(x).$$

Then $T_\infty = V_0 + V$. In the following, we study V only, since we easily see that V_0 is bounded on L_w^1 ($w \in A_1$).

We set (see [5, 6, 7])

$$B_i = \sum_{|Q|=2^{in}} b_Q \quad (i \geq 1), \quad B_0 = \sum_{|Q| \leq 1} b_Q.$$

Put $\mathcal{U} = \cup \tilde{Q}$, where \tilde{Q} denotes the cube with the same center as Q and with sidelength 100 times that of Q . Here and in the sequel all cubes we consider have sides parallel to the coordinate axes.

When $x \in \mathbf{R}^n \setminus \mathcal{U}$, we observe that

$$\begin{aligned} (4.10) \quad V(b)(x) &= V \left(\sum_{i \geq 0} B_i \right) (x) \\ &= \sum_{i \geq 0} \sum_{j \geq 1} \int K_j(x, y) B_i(y) dy = \sum_{i \geq 0} \sum_{j \geq i+1} \int K_j(x, y) B_i(y) dy \\ &= \sum_{s \geq 1} \sum_{j \geq s} \int K_j(x, y) B_{j-s}(y) dy = \sum_{s \geq 1} \sum_{j \geq s} V_j(B_{j-s})(x). \end{aligned}$$

In §5 we shall prove the following.

Lemma 2. *Suppose $w \in A_1$. There exists an $\epsilon > 0$ such that, for any positive integer s ,*

$$\left\| \sum_{j \geq s} V_j(B_{j-s}) \right\|_{L_w^2}^2 \leq c 2^{-\epsilon s} \lambda \|f\|_{L_w^1}.$$

In §6 we shall prove the following.

Lemma 3. *Suppose $w \in A_1$. Let $\|\cdot\|_{2,w}$ denote the operator norm on L_w^2 . Then, there exist constants $c, \delta > 0$ such that*

$$\|V_j\|_{2,w} \leq c 2^{-\delta j} \quad \text{for all } j \geq 1.$$

Assuming Lemmas 2 and 3, we now prove Proposition. From Lemma 3 we easily see that V is bounded on L_w^2 . By this boundedness, (4.1), (4.4), (4.5), (4.10) and Lemma 2 we have

$$\begin{aligned} (4.11) \quad w(\{x \in \mathbf{R}^n \setminus \mathcal{U} : |V(f)(x)| > \lambda\}) &\leq w(\{x \in \mathbf{R}^n \setminus \mathcal{U} : |V(g)(x)| > \lambda/2\}) + w(\{x \in \mathbf{R}^n \setminus \mathcal{U} : |V(b)(x)| > \lambda/2\}) \\ &\leq c \lambda^{-2} \|g\|_{L_w^2}^2 + c \lambda^{-2} \left\| \sum_{s \geq 1} \sum_{j \geq s} V_j(B_{j-s}) \right\|_{L_w^2}^2 \\ &\leq c \lambda^{-1} \|f\|_{L_w^1} + c \lambda^{-2} \left(\sum_{s \geq 1} \lambda^{1/2} 2^{-\epsilon s/2} \|f\|_{L_w^1}^{1/2} \right)^2 \leq c \lambda^{-1} \|f\|_{L_w^1}. \end{aligned}$$

On the other hand, by (4.3) we see that

$$(4.12) \quad w(\mathcal{U}) \leq c_w \lambda^{-1} \|f\|_{L_w^1}.$$

Combining (4.11) and (4.12), we get the boundedness of V from L_w^1 to $L_w^{1,\infty}$. This completes the proof of Proposition.

5. PROOF OF LEMMA 2

In this section we shall prove Lemma 2 in §4. For $k, m \geq 1$, put

$$(5.1) \quad H_{km}(x, y) = \int \overline{K}_k(z, x) K_m(z, y) dz \\ = \int e^{-iP(z, x) + iP(z, y)} \overline{K}(z - x) K(z - y) \varphi_k(z - x) \varphi_m(z - y) dz.$$

Then $V_k^* V_m(f)(x) = \int H_{km}(x, y) f(y) dy$, where V_k^* denotes the adjoint of V_k .

Lemma 4. *Let $k \geq m \geq 1$. Then, $H_{km}(x, y) = 0$ unless $|x - y| \leq 42^k$; and*

$$(1) \quad |H_{km}(x, y)| \leq c 2^{-kn}, \\ (2) \quad |H_{km}(x, y)| \leq c 2^{-kn} 2^{-m} |q(x) - q(y)|^{-1/M}.$$

Proof. We prove the estimate (2) only since the other assertions immediately follow from the definition of H_{km} in (5.1). We first note that

$$(\partial/\partial z_1)^M (P(z, x) - P(z, y)) = M! (q(x) - q(y)).$$

Hence, from van der Corput's lemma it follows that

$$\left| \int_a^b e^{i(P(z, x) - P(z, y))} dz_1 \right| \leq c |q(x) - q(y)|^{-1/M},$$

for any a and b (see [2, p.152]).

Therefore by integration by parts in variable z_1 in the formula of (5.1), and by using the estimates in (1.1), we easily get the conclusion.

For the rest of this note $P(x)$ will denote a real-valued polynomial on \mathbf{R}^n .

Definition 1. For a polynomial $P(x) = \sum_{|\alpha| \leq N} a_\alpha x^\alpha$ of degree N , define

$$\|P\| = \max_{|\alpha|=N} |a_\alpha|.$$

Definition 2. For a polynomial P and $\beta > 0$, let

$$\mathcal{R}(P, \beta) = \{x \in \mathbf{R}^n : |P(x)| \leq \beta\}.$$

Let $d(E, F)$ denote the distance between sets E and F . We now state a geometrical lemma for polynomials, which will be proved in §7.

Lemma 5. *Let k, m be integers such that $k \geq m$. Suppose $N \geq 1$. Then, for any polynomial P of degree N satisfying $\|P\| = 1$ and any $\gamma > 0$, there exists a positive constant $C_{n, N, \gamma}$ depending only on n, N and γ such that*

$$|\{x \in B(a, 2^k) : d(x, \mathcal{R}(P, 2^{N^m})) \leq \gamma 2^m\}| \leq C_{n, N, \gamma} 2^{(n-1)k} 2^m$$

uniformly in $a \in \mathbf{R}^n$.

Let $\lambda > 0$ and let $\{\mathcal{B}_j\}_{j \geq 0}$ be a family of measurable functions such that

$$(5.2) \quad \int_Q |\mathcal{B}_j| \leq \lambda |Q|$$

for all cubes Q in \mathbf{R}^n with sidelength $\ell(Q) = 2^j$.

Then we have the following.

Lemma 6. *Let the kernels H_{ji} be as in Lemma 4. Then, we can find a constant c such that*

$$\sum_{i=s}^j \sup_{x \in \mathbf{R}^n} \left| \int \mathcal{B}_{i-s}(y) H_{ji}(x, y) dy \right| \leq c \lambda 2^{-s}$$

for all integers j and s such that $0 < s \leq j$.

Definition 3. For $m \in \mathbf{Z}$ (the set of all integers), let \mathcal{D}_m be the family of all closed dyadic cubes Q with sidelength $\ell(Q) = 2^m$.

Proof of Lemma 6. Fix $x \in \mathbf{R}^n$. Let

$$\mathcal{F} = \{Q \in \mathcal{D}_{i-s} : Q \cap B(x, 2^{j+2}) \neq \emptyset\} \quad (0 < s \leq i \leq j).$$

Then clearly $\sum_{Q \in \mathcal{F}} |Q| \leq c 2^{jn}$.

Decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where

$$\mathcal{F}_0 = \left\{ Q \in \mathcal{F} : Q \cap \mathcal{R}(q(\cdot) - q(x), 2^{L(i-s)}) \neq \emptyset \right\}$$

and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. Then by Lemma 5 we have

$$(5.3) \quad \sum_{Q \in \mathcal{F}_0} |Q| \leq c 2^{(n-1)j} 2^{i-s}.$$

By Lemma 4 (1), (5.2) and (5.3), we see that

$$(5.4) \quad \begin{aligned} \sum_{Q \in \mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &\leq c 2^{-jn} \sum_{Q \in \mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y)| dy \\ &\leq c 2^{-jn} \lambda \sum_{Q \in \mathcal{F}_0} |Q| \\ &\leq c 2^{-jn} \lambda 2^{(n-1)j} 2^{i-s} \\ &= c \lambda 2^{i-j-s}. \end{aligned}$$

Next, by Lemma 4 (2), (5.2) and the estimate $\sum_{Q \in \mathcal{F}_1} |Q| \leq c 2^{jn}$, we have

$$(5.5) \quad \begin{aligned} \sum_{Q \in \mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &\leq c 2^{-jn} 2^{-i} 2^{-L(i-s)/M} \sum_{Q \in \mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y)| dy \\ &\leq c 2^{-jn} 2^{-i} 2^{-L(i-s)/M} \lambda \sum_{Q \in \mathcal{F}_1} |Q| \\ &\leq c \lambda 2^{-i} 2^{-L(i-s)/M}. \end{aligned}$$

From (5.4) and (5.5) it follows that

$$\begin{aligned} \int |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &= \sum_{Q \in \mathcal{F}} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy \\ &= \sum_{\nu=0}^1 \sum_{Q \in \mathcal{F}_\nu} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy \\ &\leq c \lambda \left(2^{i-j-s} + 2^{-i} 2^{-L(i-s)/M} \right). \end{aligned}$$

Thus we see that

$$\begin{aligned} \sum_{i=s}^j \sup_{x \in \mathbf{R}^n} \int |\mathcal{B}_{i-s}(y) H_{ji}(x, y)| dy &\leq c\lambda \sum_{i=s}^j \left(2^{i-j-s} + 2^{-i} 2^{-L(i-s)/M} \right) \\ &\leq c\lambda 2^{-s}. \end{aligned}$$

This completes the proof of Lemma 6.

By Lemma 6 we readily get the following.

Lemma 7. *Let $\{\mathcal{B}_j\}_{j \geq 0}$ be as in Lemma 6. Suppose $\sum_{j \geq 0} \|\mathcal{B}_j\|_{L^1} < \infty$. Then, for any positive integer s , we have*

$$\left\| \sum_{j \geq s} V_j(\mathcal{B}_{j-s}) \right\|_{L^2}^2 \leq c\lambda 2^{-s} \sum_{j \geq 0} \|\mathcal{B}_j\|_{L^1}.$$

Proof. Let $\langle \cdot, \cdot \rangle$ denote the inner product in L^2 . Using Lemma 6, we see that

$$\begin{aligned} \left\| \sum_{j \geq s} V_j(\mathcal{B}_{j-s}) \right\|_{L^2}^2 &\leq 2 \sum_{j \geq s} \sum_{i=s}^j |\langle V_j(\mathcal{B}_{j-s}), V_i(\mathcal{B}_{i-s}) \rangle| \\ &\leq 2 \sum_{j \geq s} \sum_{i=s}^j |\langle \mathcal{B}_{j-s}, V_j^* V_i(\mathcal{B}_{i-s}) \rangle| \\ &\leq 2 \sum_{j \geq s} \sum_{i=s}^j \|\mathcal{B}_{j-s}\|_{L^1} \|V_j^* V_i(\mathcal{B}_{i-s})\|_{L^\infty} \\ &\leq c\lambda 2^{-s} \sum_{j \geq s} \|\mathcal{B}_{j-s}\|_{L^1}. \end{aligned}$$

This completes the proof of Lemma 7.

Definition 4. For each $j \geq 0$, let \mathcal{G}_j be a family of non-overlapping closed dyadic cubes Q such that $\ell(Q) \leq 2^j$. We suppose that if $Q \in \mathcal{G}_j$, $R \in \mathcal{G}_k$ and $j \neq k$, then Q and R are non-overlapping and that $\sum_{j \geq 0} \sum_{Q \in \mathcal{G}_j} |Q| < \infty$. Put $\mathcal{G} = \cup_{j \geq 0} \mathcal{G}_j$.

Let $\lambda > 0$. To each $Q \in \mathcal{G}$ we associate $f_Q \in L^1$ such that

$$\int |f_Q| \leq \lambda |Q|, \quad \text{supp}(f_Q) \subset Q.$$

We define

$$\mathcal{A}_i = \sum_{Q \in \mathcal{G}_i} f_Q.$$

Lemma 8. *Let v be a locally integrable positive function and let s be a positive integer. Then*

$$\left\| \sum_{j \geq s} V_j(\mathcal{A}_{j-s}) \right\|_{L_v^2}^2 \leq c\lambda^2 \sum_{Q \in \mathcal{G}} |Q| \inf_Q M M(v),$$

where $\inf_Q f = \inf_{x \in Q} f(x)$.

Proof. The proof we give here is essentially the same as that in [11]. We include the proof for the sake of completeness. We may assume $\sum_{Q \in \mathcal{G}} |Q| \inf_Q M M(v) < \infty$. Let $\langle \cdot, \cdot \rangle_v$ denote the inner product in L_v^2 . Then, if $s \leq i \leq j$, we see that

$$\begin{aligned} & \langle V_j(\mathcal{A}_{j-s}), V_i(\mathcal{A}_{i-s}) \rangle_v \\ &= \int \left(\int K_j(x, y) \mathcal{A}_{j-s}(y) dy \int \overline{K}_i(x, z) \overline{\mathcal{A}}_{i-s}(z) dz \right) v(x) dx \\ &= \int \mathcal{A}_{j-s}(y) \int \overline{\mathcal{A}}_{i-s}(z) \left(\int K_j(x, y) \overline{K}_i(x, z) v(x) dx \right) dz dy. \end{aligned}$$

Put

$$v(y, z; i, j) = 2^{-in} 2^{-jn} \int_{B(y, 2^{j+2}) \cap B(z, 2^{i+2})} v(x) dx.$$

Let c_Q denote the center of a cube Q . If $Q \in \mathcal{G}_{j-s}$, $R \in \mathcal{G}_{i-s}$ and if $B(y, 2^{j+2})$ intersects $B(z, 2^{i+2})$ for some $y \in Q$ and some $z \in R$, then $R \subset B(c_Q, n^{1/2} 2^{j+10})$. Thus we have

$$\begin{aligned} & \sum_{i=s}^j |\langle V_j(\mathcal{A}_{j-s}), V_i(\mathcal{A}_{i-s}) \rangle_v| \\ & \leq c \sum_{i=s}^j \int |\mathcal{A}_{j-s}(y)| \int |\mathcal{A}_{i-s}(z)| v(y, z; i, j) dz dy \\ & \leq c \sum_{Q \in \mathcal{G}_{j-s}} \int |f_Q(y)| \sum_{i=s}^j \sum_{R \in \mathcal{G}_{i-s}} \int |f_R(z)| v(y, z; i, j) dz dy \\ & \leq c \sum_{Q \in \mathcal{G}_{j-s}} 2^{-jn} \int |f_Q(y)| dy \sum_{i=s}^j \sum_{\substack{R \in \mathcal{G}_{i-s} \\ R \subset B(c_Q, n^{1/2} 2^{j+10})}} \inf_R M(v) \int |f_R(z)| dz \\ & = I, \end{aligned}$$

say. Since

$$\inf_R M(v) \int |f_R(z)| dz \leq \lambda |R| \inf_R M(v) \leq \lambda \int_R M(v)(z) dz$$

and cubes in \mathcal{G} are non-overlapping,

$$\begin{aligned}
I &\leq c\lambda \sum_{Q \in \mathcal{G}_{j-s}} 2^{-jn} \int_{B(c_Q, n^{1/2}2^{j+10})} M(v)(z) dz \int |f_Q(y)| dy \\
&\leq c\lambda \sum_{Q \in \mathcal{G}_{j-s}} \inf_Q MM(v) \int |f_Q(y)| dy \\
&\leq c\lambda^2 \sum_{Q \in \mathcal{G}_{j-s}} |Q| \inf_Q MM(v).
\end{aligned}$$

Therefore, we get the conclusion by summing over $j \geq s$, since

$$\left\| \sum_{j \geq s} V_j(\mathcal{A}_{j-s}) \right\|_{L_v^2}^2 \leq 2 \sum_{j \geq s} \sum_{i=s}^j |\langle V_j(\mathcal{A}_{j-s}), V_i(\mathcal{A}_{i-s}) \rangle_v|.$$

We prove Lemma 2 by the interpolation argument of [11] between the estimates of Lemma 7 and Lemma 8.

Lemma 9. *Let \mathcal{F} denote the family of dyadic cubes arising from the Calderón-Zygmund decomposition in §4. Then, for all $t > 0$, we have*

$$\int \left| \sum_{j \geq s} V_j(B_{j-s})(x) \right|^2 \min(v(x), t) dx \leq c\lambda^2 \sum_{Q \in \mathcal{F}} |Q| \min \left(t2^{-s}, \inf_Q MM(v) \right),$$

where s is a positive integer and v is a locally integrable positive function.

Proof. We define

$$\mathcal{F}_t = \left\{ Q \in \mathcal{F} : \inf_Q MM(v) < t2^{-s} \right\}$$

and $\mathcal{F}_t^* = \mathcal{F} \setminus \mathcal{F}_t$. For $j \geq 1$, put

$$B'_j = \sum_{\substack{|Q|=2^{jn} \\ Q \in \mathcal{F}_t}} b_Q, \quad B''_j = \sum_{\substack{|Q|=2^{jn} \\ Q \in \mathcal{F}_t^*}} b_Q$$

and

$$B'_0 = \sum_{\substack{|Q| \leq 1 \\ Q \in \mathcal{F}_t}} b_Q, \quad B''_0 = \sum_{\substack{|Q| \leq 1 \\ Q \in \mathcal{F}_t^*}} b_Q.$$

Then $B_j = B'_j + B''_j$ for $j \geq 0$. Hence

$$\begin{aligned}
& \int \left| \sum_{j \geq s} V_j(B_{j-s})(x) \right|^2 \min(v(x), t) dx \\
& \leq 2 \int \left| \sum_{j \geq s} V_j(B'_{j-s})(x) \right|^2 \min(v(x), t) dx + 2 \int \left| \sum_{j \geq s} V_j(B''_{j-s})(x) \right|^2 \min(v(x), t) dx \\
& \leq 2 \int \left| \sum_{j \geq s} V_j(B'_{j-s})(x) \right|^2 v(x) dx + 2 \int \left| \sum_{j \geq s} V_j(B''_{j-s})(x) \right|^2 t dx \\
& = I + II,
\end{aligned}$$

say.

Applying Lemma 8 with $\mathcal{A}_j = c_1 B'_j$ (see (4.7) and (4.9)), we get

$$I \leq c\lambda^2 \sum_{Q \in \mathcal{F}_t} |Q| \inf_Q MM(v) = c\lambda^2 \sum_{Q \in \mathcal{F}_t} |Q| \min \left(t2^{-s}, \inf_Q MM(v) \right).$$

By Lemma 7 with $\mathcal{B}_j = c_2 B''_j$ (see (4.7) and (4.9)), we have

$$\begin{aligned}
II & \leq c\lambda t 2^{-s} \sum_{j \geq 0} \|B''_j\|_{L^1} \\
& \leq c\lambda^2 t 2^{-s} \sum_{Q \in \mathcal{F}_t^*} |Q| \\
& = c\lambda^2 \sum_{Q \in \mathcal{F}_t^*} |Q| \min \left(t2^{-s}, \inf_Q MM(v) \right).
\end{aligned}$$

(Here c_1 and c_2 are normalizing constants.) Combining the estimates for I and II , we get the conclusion.

Now we finish the proof of Lemma 2. Multiplying both sides of the inequality in Lemma 9 by $t^{-\theta}$ ($\theta \in (0, 1)$), then integrating them on $(0, \infty)$ with respect to the measure dt/t and using

$$\int_0^\infty \min(A, t) t^{-\theta} \frac{dt}{t} = c_\theta A^{1-\theta} \quad (A > 0) \quad \text{for some } c_\theta > 0,$$

we get

(5.6)

$$\begin{aligned}
\int \left| \sum_{j \geq s} V_j(B_{j-s})(x) \right|^2 v(x)^{1-\theta} dx & \leq c\lambda^2 \sum_{Q \in \mathcal{F}} |Q| 2^{-\theta s} \inf_Q MM(v)^{1-\theta} \\
& \leq c\lambda 2^{-\theta s} \sum_{Q \in \mathcal{F}} \int_Q |f(x)| dx \inf_Q MM(v)^{1-\theta} \\
& \leq c\lambda 2^{-\theta s} \int |f(x)| MM(v)(x)^{1-\theta} dx,
\end{aligned}$$

where the second inequality follows from (4.2).

If $w \in A_1$, then $w^{1+\delta} \in A_1$ for some $\delta > 0$; so substituting $w^{1+\delta}$ for v and putting $\theta = \delta/(1+\delta)$ in (5.6), we get Lemma 2.

6. PROOF OF LEMMA 3

In this section we shall prove Lemma 3 in §4.

Lemma 10. *Let $\|\cdot\|_2$ denote the operator norm on L^2 . Then, for $j \geq 1$,*

$$\|V_j\|_2 \leq \begin{cases} C_{M,L} 2^{-j/2 - \min(L/M, M/L)j/2} & (M \neq L) \\ C_M j^{1/2} 2^{-j} & (M = L). \end{cases}$$

Estimates of this kind have been obtained in Ricci-Stein [9]. Here we give an alternative proof.

Proof of Lemma 10. Fix x . Let

$$E = \mathcal{R}(q(\cdot) - q(x), 2^{-jM}) \cap B(x, 2^{j+2}) \quad \text{and} \quad F = B(x, 2^{j+2}) \setminus E.$$

Define

$$\mathcal{E} = \{Q \in \mathcal{D}_{-[jM/L]} : Q \cap E \neq \emptyset\},$$

where $[a]$ denotes the greatest integer not exceeding a .

Then, by Lemma 4 (1) and Lemma 5 we have

$$\begin{aligned} (6.1) \quad \int_E |H_{jj}(x, y)| dy &\leq \sum_{Q \in \mathcal{E}} \int_Q |H_{jj}(x, y)| dy \\ &\leq c 2^{-jn} \sum_{Q \in \mathcal{E}} |Q| \\ &\leq c 2^{-j} 2^{-jM/L}. \end{aligned}$$

For $\nu = 0, 1, 2, \dots$, let

$$F_\nu = B(x, 2^{j+2}) \cap (\mathcal{R}(q(\cdot) - q(x), 2^{-jM+\nu+1}) \setminus \mathcal{R}(q(\cdot) - q(x), 2^{-jM+\nu})).$$

Then $F = \cup_{\nu=0}^\infty F_\nu$. For $0 \leq \nu \leq j(M+L)-1$, let

$$\mathcal{F}_\nu = \{Q \in \mathcal{D}_{-[(jM-\nu-1)/L]} : Q \cap F_\nu \neq \emptyset\}.$$

Then by Lemma 5 we have

$$|F_\nu| \leq \sum_{Q \in \mathcal{F}_\nu} |Q| \leq c 2^{j(n-1)} 2^{-[(jM-\nu-1)/L]}.$$

So by Lemma 4 (2) we see that

$$\begin{aligned} \int_{F_\nu} |H_{jj}(x, y)| dy &\leq c 2^{-jn} 2^{-j} 2^{j-\nu/M} |F_\nu| \\ &\leq c 2^{-j} 2^{-\nu/M} 2^{-[(jM-\nu-1)/L]} \\ &\leq c 2^{-j} 2^{-jM/L} 2^{-\nu(1/M-1/L)}. \end{aligned}$$

Thus, if $M \neq L$,

$$\begin{aligned}
 (6.2) \quad & \sum_{\nu=0}^{j(M+L)-1} \int_{F_\nu} |H_{jj}(x, y)| \, dy \\
 & \leq c 2^{-j} 2^{-jM/L} (2^{-(1/M-1/L)} - 1)^{-1} (2^{-j(M+L)(1/M-1/L)} - 1) \\
 & = c 2^{-j} (2^{-(1/M-1/L)} - 1)^{-1} (2^{-jL/M} - 2^{-jM/L});
 \end{aligned}$$

and if $M = L$,

$$(6.3) \quad \sum_{\nu=0}^{j(M+L)-1} \int_{F_\nu} |H_{jj}(x, y)| \, dy \leq c 2^{-2j} j(M+L).$$

Finally, by Lemma 4 (2) we have

$$\begin{aligned}
 (6.4) \quad & \int_{\cup_{\nu \geq j(M+L)} F_\nu} |H_{jj}(x, y)| \, dy \leq c 2^{-jn} 2^{-j} 2^{-jL/M} \left| \bigcup_{\nu \geq j(M+L)} F_\nu \right| \\
 & \leq c 2^{-j} 2^{-jL/M}.
 \end{aligned}$$

By (6.1), (6.2), (6.3) and (6.4) we see that

$$\sup_{x \in \mathbf{R}^n} \int |H_{jj}(x, y)| \, dy \leq \begin{cases} C_{M,L} 2^{-j - \min(L/M, M/L)j} & (M \neq L) \\ C_M j 2^{-2j} & (M = L). \end{cases}$$

We have the same estimate for $\sup_y \int |H_{jj}(x, y)| \, dx$. From these results we get the conclusion since

$$\|V_j^* V_j(f)\|_{L^2} \leq \left(\sup_x \int |H_{jj}(x, y)| \, dy \right)^{1/2} \left(\sup_y \int |H_{jj}(x, y)| \, dx \right)^{1/2} \|f\|_{L^2}$$

and $\|V_j^* V_j\|_2 = \|V_j\|_2^2$.

Now we prove Lemma 3. It is easy to see that $\|V_j\|_{2,w} \leq c_w$. By interpolation with change of measure between this estimate and that of Lemma 10, we get

$$(6.5) \quad \|V_j\|_{2,w^\theta} \leq c_{w,\theta} 2^{-(1-\theta)j/2}$$

for all $\theta \in (0, 1)$. We have $w^{1+\epsilon} \in A_1$ for some $\epsilon > 0$; so substituting $w^{1+\epsilon}$ for w and putting $\theta = 1/(1+\epsilon)$ in (6.5), we get the desired estimate.

7. PROOF OF LEMMA 5

Our proof is an application of the methods appearing in the proof of [2, LEMMA 4.1]. We use some tools and results given in [2].

Definition 5. Suppose $n \geq 2$. Let

$$S_m = \{Q_m + (0, 0, \dots, 0, j) : j \in \mathbf{Z}\},$$

where $m = (m_1, m_2, \dots, m_{n-1}) \in \mathbf{Z}^{n-1}$ and $Q_m = [0, 1]^n + (m_1, m_2, \dots, m_{n-1}, 0)$. We call S_m a strip.

Definition 6. Suppose $n \geq 2$. For $m \in \mathbf{Z}^{n-1}$, we define

$$I_m = \{Q_m + (0, 0, \dots, 0, j) : j_1 < j < j_2\},$$

where $j_1, j_2 \in \mathbf{Z} \cup \{-\infty, \infty\}$ and Q_m is as in Definition 5. We call I_m an interval.

Definition 7. For a set $E \subset \mathbf{R}^n$, we put

$$\mathcal{N}(E) = \{x \in \mathbf{R}^n : d(x, E) \leq 1\}.$$

Let P be a polynomial of degree N as in Lemma 5. We consider $\mathcal{R}(P, \beta)$ for $\beta > 0$ (see Definition 2).

Lemma 11. Suppose that $n \geq 2$ and $N \geq 1$. There exists a positive integer $C_{n,N}$ depending only on n and N such that for $i = 1, 2, \dots, C_{n,N}$ we can find $U_i \in O(n)$ (the orthogonal group) and families of cubes $J_{m,i} \subset S_m$ ($m \in \mathbf{Z}^{n-1}$) so that

$$(1) \quad \mathcal{N}(\mathcal{R}(P, \beta)) \subset \bigcup_{i=1}^{C_{n,N}} U_i(\mathcal{L}_i), \text{ where}$$

$$\mathcal{L}_i = \bigcup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_{m,i} \right\};$$

$$(2) \quad \text{card}(J_{m,i}) \leq c \text{ for some constant } c \text{ depending only on } n, N \text{ and } \beta.$$

Remark 3. If Lemma 11 holds, then we have, for any $\gamma > 0$,

$$\{x : d(x, \mathcal{R}(P, \beta)) \leq \gamma\} \subset \bigcup_{i=1}^{C_{n,N,\gamma}} U_i(\mathcal{L}_i)$$

for some positive integer $C_{n,N,\gamma}$ depending only on n, N and γ , where U_i and \mathcal{L}_i are as in Lemma 11. This can be proved by considering a finite number of polynomials which are defined by translating P and by applying Lemma 11 to each of them. (See [2, p. 149].)

To prove Lemma 11, we need the following results given in [2].

Sublemma 1. Suppose $n \geq 2$. For any positive integer N , there exists a positive integer $C_{n,N}$ depending only on n and N such that for any strip S , any polynomial P of degree N and any $\gamma > 0$

$$\{Q \in S : Q \cap \mathcal{R}(P, \gamma) \neq \emptyset\}$$

is a union of at most $C_{n,N}$ intervals. (See LEMMA 4.2 of [2].)

Sublemma 2. *Suppose $n \geq 2$. For any positive integer N , there exist positive constants $A_{n,N}$ and $B_{n,N}$ depending only on n and N such that*

$$A_{n,N} \|P\| \leq \|P \circ \Xi\| \leq B_{n,N} \|P\|$$

for every polynomial P of degree N and every $\Xi \in O(n)$, where $P \circ \Xi(x) = P(\Xi x)$.

Sublemma 3. *Suppose $n \geq 2$. For any positive integer N , there exists a positive constant $C_{n,N}$ depending only on n and N such that for any polynomial P of degree N we can find $\Theta \in O(n)$ so that*

$$\min_{1 \leq j \leq n} \|D_j(P \circ \Theta)\| \geq C_{n,N} \|P \circ \Theta\|,$$

where $D_j = \partial/\partial x_j$.

Now we prove Lemma 11. We use induction on the polynomial degree N . Let $A(N)$ be the assertion of Lemma 11 for polynomials of degree N .

Proof of $A(1)$. Let $P(x) = \sum_{i=1}^n a_i x_i + b$. First, we consider the case $|a_n| = 1$. Now we show that if I is an interval such that each cube of I intersects $\mathcal{R}(P, \beta)$, then $\text{card}(I) \leq c$ for some c depending only on n and β . Let $y \in Q \in I$ satisfy $|P(y)| \leq \beta$. We note that

$$P(y + de_n) - P(y) = da_n \quad \text{for } d \in \mathbf{R},$$

where e_j is the element of \mathbf{R}^n whose j th coordinate is 1 and whose other coordinates are all 0. Therefore, if $y + de_n \in Q' \in I$, we see that

$$\inf_{z \in Q'} |P(z)| \geq |P(y + de_n)| - \sum_{i=1}^n |a_i| \geq |da_n| - \beta - \sum_{i=1}^n |a_i| \geq |d| - \beta - n.$$

This easily implies that $\text{card}(I) \leq c$.

By this and Sublemma 1, there exists a constant c depending only on n and β such that

$$\text{card}(\{Q \in S : Q \cap \mathcal{R}(P, \beta) \neq \emptyset\}) \leq c$$

for all strips S .

Therefore, if we put

$$J_m = \{Q \in S_m : d(Q, \mathcal{R}(P, \beta)) \leq 1\},$$

then $\text{card}(J_m) \leq c$ for some c depending only on n and β ; and $\mathcal{N}(\mathcal{R}(P, \beta)) \subset \mathcal{L}$, where

$$\mathcal{L} = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_m \right\}.$$

Next, we consider any polynomial P of degree 1 such that $\|P\| = 1$. Then if $P_1(x) = P(Ux)$ for a suitable $U \in O(n)$, we have $D_n P_1 = 1$. Hence, by what we have already proved we get $\mathcal{N}(\mathcal{R}(P_1, \beta)) \subset \mathcal{L}$. It follows that $\mathcal{N}(\mathcal{R}(P, \beta)) \subset U(\mathcal{L})$ since $\mathcal{N}(\mathcal{R}(P \circ U, \beta)) = U^{-1} \mathcal{N}(\mathcal{R}(P, \beta))$. This completes the proof of $A(1)$.

Now we assume $A(N-1)$ ($N \geq 2$) and prove $A(N)$. For a polynomial P of degree N such that $\|P\| = 1$, we choose $\Theta \in O(n)$ as in Sublemma 3. Put

$$E_0 = \mathcal{R}(P \circ \Theta, \beta) \cap \left(\bigcup_{j=1}^n \mathcal{R}(D_j(P \circ \Theta), \beta) \right);$$

and for $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \in \{-1, 1\}^n$ put

$$E_\kappa = \{x \in \mathcal{R}(P \circ \Theta, \beta) : \kappa_j D_j(P \circ \Theta)(x) > \beta \text{ for } j = 1, 2, \dots, n\}.$$

Then

$$\mathcal{R}(P \circ \Theta, \beta) = E_0 \cup \left(\bigcup_{\kappa \in \{-1, 1\}^n} E_\kappa \right)$$

and so

$$(7.1) \quad \mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \mathcal{N}(E_0) \cup \left(\bigcup_{\kappa \in \{-1, 1\}^n} \mathcal{N}(E_\kappa) \right).$$

We separately treat the $2^n + 1$ sets on the right hand side.

First, clearly

$$(7.2) \quad \mathcal{N}(E_0) \subset \bigcup_{j=1}^n \mathcal{N}(\mathcal{R}(D_j(P \circ \Theta), \beta)).$$

Since $C_j = \|D_j(P \circ \Theta)\| \sim 1$ (this means that $c^{-1} \leq \|D_j(P \circ \Theta)\| \leq c$ for some $c > 1$ depending only on n and N) and $\mathcal{R}(D_j(P \circ \Theta), \beta) = \mathcal{R}(C_j^{-1} D_j(P \circ \Theta), C_j^{-1} \beta)$, we can apply the induction hypothesis $A(N-1)$ to the right hand side of (7.2).

Next, we fix κ and consider $\mathcal{N}(E_\kappa)$. Pick $O_\kappa \in O(n)$ such that $O_\kappa(e_n) = n^{-1/2} \kappa$. Define

$$\mathcal{D}_0^* = \mathcal{D}_0 \setminus \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^n \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}.$$

Since $\|(D_j(P \circ \Theta)) \circ O_\kappa\| \sim 1$ by Sublemmas 2 and 3, we can apply the hypothesis $A(N-1)$ along with Remark 3 to

$$G = \bigcup \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^n \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}$$

to get

$$(7.3) \quad \mathcal{N}(G) \subset \bigcup_i U'_i(\mathcal{L}'_i)$$

for some $U'_i \in O(n)$ and some \mathcal{L}'_i such that

$$\mathcal{L}'_i = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J'_{m,i} \right\}$$

for some $J'_{m,i} (\subset S_m)$ satisfying $\text{card}(J'_{m,i}) \leq c$.

We have to study $O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)$. First, we note that if $O_\kappa^{-1}(E_\kappa)$ intersects Q , $Q \in \mathcal{D}_0^*$, then

$$(7.4) \quad \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all } y \in Q.$$

This can be seen as follows. Suppose that there are j_0 and $y_0 \in Q$ such that $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_\kappa y_0) \leq \beta$. Then, since we have $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_\kappa x) > \beta$ for some $x \in Q$, by the intermediate value theorem we can find $z \in Q$ such that $|D_{j_0}(P \circ \Theta)(O_\kappa z)| \leq \beta$. This contradicts the fact that $Q \in \mathcal{D}_0^*$.

By (7.4) we have

$$(7.5) \quad O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*) \subset \cup \left\{ Q \in \mathcal{D}_0 : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \right\}.$$

For a strip S , put

$$\mathcal{E} = \left\{ Q \in S : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \right\}.$$

We shall show $\text{card}(\mathcal{E}) \leq C_{n,N}$.

We first see that \mathcal{E} is a union of at most $C_{n,N}$ intervals. Put

$$\mathcal{E}' = \left\{ Q \in S : \min_{1 \leq j \leq n} |D_j(P \circ \Theta)(O_\kappa y)| > \beta \quad \text{for all } y \in Q \right. \\ \left. \text{and } \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \right\}.$$

Then

$$\mathcal{E}' = \left(\bigcap_{j=1}^n (S \setminus \{Q \in S : \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \cap Q \neq \emptyset\}) \right) \\ \cap \{Q \in S : \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset\}.$$

We observe that the complement of a finite union of intervals in a strip S is also a finite union of intervals, and the intersection of finite unions of intervals is also a finite union

of intervals. Hence, by Sublemma 1 we see that \mathcal{E}' is a union of at most $C_{n,N}$ intervals : $\mathcal{E}' = \cup_i J_i$.

Consider any J_i . Then by the intermediate value theorem we have either

$$\min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all } y \in \cup \{Q : Q \in J_i\}$$

or

$$\min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) < -\beta \quad \text{for all } y \in \cup \{Q : Q \in J_i\}.$$

Thus \mathcal{E} is a union of a subfamily $\{I_i\}$ of $\{J_i\}$: $\mathcal{E} = \cup_i I_i$.

Let I be any interval in $\{I_i\}$. We need the following (see [2, p. 151]).

Sublemma 4. *There exists a constant c_n depending only on n such that if $x, y \in I$ and $y_n - x_n \geq c_n$, then*

$$y - x = \sum_{i=1}^n \lambda_i O_\kappa^{-1} e_i$$

for some $\lambda_i \in \mathbf{R}$ such that $\kappa_i \lambda_i \geq 3$.

Proof. We see that

$$\begin{aligned} O_\kappa(y - x) &= \sum_{i=1}^n (y_i - x_i) O_\kappa e_i = \sum_{i=1}^{n-1} (y_i - x_i) O_\kappa e_i + (y_n - x_n) n^{-1/2} \kappa \\ &= \sum_{i=1}^n \left(n^{-1/2} (y_n - x_n) \kappa_i + b_i \right) e_i \end{aligned}$$

for some $b_i \in \mathbf{R}$ such that $|b_i| \leq c$, which is feasible since $|y_i - x_i| \leq 1$ for $i = 1, 2, \dots, n-1$. This readily implies the conclusion.

Put $Y = P \circ \Theta \circ O_\kappa$. Then $\nabla Y(x) = O_\kappa^{-1}(\nabla(P \circ \Theta)(O_\kappa x))$; so, if $x, y \in I$ and $y_n - x_n \geq c_n$, by Sublemma 4 we have

$$\begin{aligned} Y(y) - Y(x) &= \int_0^1 \langle y - x, (\nabla Y)(x + t(y - x)) \rangle dt \\ &= \int_0^1 \sum_{i=1}^n \lambda_i \langle O_\kappa^{-1} e_i, O_\kappa^{-1} (\nabla(P \circ \Theta)(O_\kappa(x + t(y - x)))) \rangle dt \\ &= \int_0^1 \sum_{i=1}^n \lambda_i D_i(P \circ \Theta)(O_\kappa(x + t(y - x))) dt \\ &\geq \sum_{i=1}^n \lambda_i \kappa_i \beta \geq 3n\beta > 3\beta, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^n . Since $\mathcal{R}(Y, \beta) \cap Q \neq \emptyset$ for all $Q \in I$, we can conclude that $\text{card}(I) \leq c_n + 3$.

Combining the above results, we have $\text{card}(\mathcal{E}) \leq C_{n,N}$ as claimed. From this and (7.5) we easily see that

$$(7.6) \quad \mathcal{N}(O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)) \subset \mathcal{L},$$

where $\mathcal{L} = \cup \{Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_m\}$ for some $J_m \subset S_m$ with $\text{card}(J_m) \leq C_{n,N}$.

By (7.3) and (7.6) we have

$$\begin{aligned} \mathcal{N}(O_\kappa^{-1}(E_\kappa)) &\subset \mathcal{N}(G) \cup \mathcal{N}(O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)) \\ &\subset (\cup_i U_i'(\mathcal{L}'_i)) \cup \mathcal{L}; \end{aligned}$$

and so, observing $\mathcal{N}(O_\kappa^{-1}(E_\kappa)) = O_\kappa^{-1}\mathcal{N}(E_\kappa)$,

$$(7.7) \quad \mathcal{N}(E_\kappa) \subset (\cup_i O_\kappa U_i'(\mathcal{L}'_i)) \cup O_\kappa(\mathcal{L}).$$

Since $\mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \Theta^{-1}\mathcal{N}(\mathcal{R}(P, \beta))$, by (7.1), (7.2) with $A(N-1)$ and (7.7) we get $A(N)$. This completes the proof of Lemma 11.

Proof of Lemma 5. We see that $\mathcal{R}(P, 2^{Nm}) = 2^m \mathcal{R}(\tilde{P}, 1)$, where

$$\tilde{P}(x) = 2^{-Nm} P(2^m x).$$

Note that $\|\tilde{P}\| = 1$. (See [2, p. 151].) This observation enables us to assume $m = 0$ to prove Lemma 5. Clearly, we may also assume $\gamma = 1$.

Thus it suffices to show, for $k \geq 0$,

$$(7.8) \quad |\{x \in B(a, 2^k) : d(x, \mathcal{R}(P, 1)) \leq 1\}| \leq C_{n,N} 2^{(n-1)k}$$

uniformly in $a \in \mathbf{R}^n$.

If $n = 1$, (7.8) easily follows from Chanillo-Christ [2, LEMMA 3.2] (see also [4]). Suppose $n \geq 2$. Then, (7.8) follows from Lemma 11 with $\beta = 1$ and the obvious estimate:

$$|B(a, 2^k) \cap U_i(\mathcal{L}_i)| \leq c 2^{(n-1)k},$$

where $U_i(\mathcal{L}_i)$ is as in Lemma 11. This completes the proof of Lemma 5.

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